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THE THERMISTOR PROBLEM: EXISTENCE, SMOOTHNESS, UNIQUENESS, BLOWUP*

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Abstract. The goal of this paper is to study a nonlinear system modeling the heat diffusion produced by Joule effect in an electric conductor. Existence, uniqueness, smoothness, and blowup in particular are studied.

Key words. parabolic systems, existence, uniqueness, smoothness, blowup

AMS subject classifications. 35K20, 35K35, 35K45, 35K60

1. Introduction. The heat produced in a conductor by an electric current leads to the so-called thermistor problem, i.e., to the system

$$u_t - \nabla \cdot (\kappa(u) \nabla u) = \sigma(u) |\nabla\varphi|^2 \quad \text{in } \Omega \times (0,T), \tag{1.1a}$$

$$u = 0 \quad \text{on } \Gamma \times (0,T), \quad u(\cdot,0) = u_0, \tag{1.1b}$$

$$\nabla \cdot (\sigma(u) \nabla\varphi) = 0 \quad \text{in } \Omega \times (0,T), \tag{1.1c}$$

$$\varphi = \varphi_0 \quad \text{on } \Gamma \times (0,T). \tag{1.1d}$$

We assume here that Ω is a smooth, bounded open set of $\mathbf{R}^n, \Gamma$ denotes its boundary, T is some positive given number, φ is the electrical potential, u the temperature inside the conductor, $\kappa(u) > 0$ the thermal conductivity, and $\sigma(u) > 0$ the electrical conductivity. The physical situation is when $n = 3$ and Ω is the spatial domain occupied by the body that we consider and which is assumed to be a conductor of both heat and electricity. However, we will consider the general case $n \geq 1$.

If $\mathcal{I}$ denotes the current density and $\mathcal{Q}$ the vector of heat flow then the Ohm law and the Fourier law read, respectively,

$$\mathcal{I} = -\sigma(u) \nabla\varphi, \tag{1.2}$$

$$\mathcal{Q} = -\kappa(u) \nabla u. \tag{1.3}$$

Then equations (1.1a) and (1.1c) follow from the conservation laws

$$\nabla \cdot \mathcal{I} = 0, \qquad \rho c \frac{\partial u}{\partial t} + \nabla \cdot \mathcal{Q} = \mathcal{I} \cdot \mathcal{E}, \tag{1.4}$$

where $\mathcal{E}$ denotes the electric field, ρ the density of the conductor, c its heat capacity (see also [C.1], [C.P.], [H.R.S.], and [Ko]). We assume here that $\rho c \equiv 1$.

Remark 1.1. Due to (1.1c), (1.1a) also reads

$$u_t = \nabla \cdot (\kappa(u) \nabla u + \sigma(u) \varphi \nabla\varphi) \quad \text{in } \Omega \times (0,T).$$

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The similarity with the two-phase filtration problem should be noticed. Indeed, if u is the concentration and φ the pressure, then the equations of two-phase filtration read

$$u_t = \nabla \cdot (\kappa(u)\nabla u + b(u)\nabla\varphi) \quad \text{in } \Omega \times (0,T),$$
$$\nabla \cdot (\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega \times (0,T).$$

We refer the reader to [A.K.M.] for details.

Instead of (1.1b) we will also consider the boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma \times (0,T), \quad u(\cdot,0) = u_0, \tag{1.1b$'$}$$

where $\partial u/\partial n$ denotes the outward normal derivative of u.

The paper is divided as follows. In §2 we will show existence of a weak solution to (1.1). In §3 we will focus on the question of smoothness. In §4 we will analyze the dependence of the solution with respect to the data and derive uniqueness results. Finally, in the last section we will investigate the issue of global existence or blowup.

We will use standard notation for parabolic problems and we refer to [L.S.U.] for details.

2. Existence of a weak solution. Let V be a subspace of $H^1(\Omega)$ containing $H_0^1(\Omega)$, V' its dual (see, for instance, [B.L.], [D.L.], [J.L.L.], or [G.T.] for the definition and the properties of the Sobolev spaces). Recall first the following well-known result of the theory of linear parabolic equations (see [D.L.], [L.S.U.]).

Assume

$$\begin{aligned} &u_0 \in L^2(\Omega), \\ &\kappa \in L^\infty(\Omega \times (0,T)), \qquad 0 < \kappa_1 \le \kappa \le \kappa_2 < +\infty, \end{aligned} \tag{2.1}$$

where κ_1, κ_2 are two positive constants.

THEOREM 2.1. *If $f \in L^2(0,T;V')$, there exists a unique u such that*

$$u \in L^2(0,T;V) \cap C([0,T];L^2(\Omega)), \qquad u_t \in L^2(0,T;V'), \tag{2.2}$$
$$\left\langle \frac{d}{dt}u, v \right\rangle + \int_\Omega \kappa \nabla u \cdot \nabla v\, dx = \langle f, v\rangle \quad \text{a.e. } t \in (0,T), \quad \forall v \in V, \tag{2.3}$$
$$u(0) = u_0. \tag{2.4}$$

Moreover, we have the estimate

$$\begin{aligned} \frac{1}{2}|u(t)|_2^2 + \kappa_1 \int_0^t \|\nabla u(t)\|_2^2\, dt &\le \frac{1}{2}|u(t)|_2^2 + \int_0^t \int_\Omega \kappa |\nabla u|^2\, dx\, dt \\ &= \frac{1}{2}|u(0)|_2^2 + \int_0^t \langle f, u(t)\rangle\, dt \quad \text{a.e. } t \in (0,T). \end{aligned} \tag{2.5}$$

($\langle\cdot\rangle$ is the duality bracket between V', V, $|\cdot|_p$ the usual L^p norm, $|\nabla u|$ the Euclidean norm of the gradient of u.)

We will assume that

$$\varphi_0 \in L^\infty(0,T;H^1(\Omega) \cap L^\infty(\Omega)), \tag{2.6}$$
$$\kappa, \sigma \text{ continuous}, \quad 0 < \kappa_1 \le \kappa \le \kappa_2, \quad 0 < \sigma_1 \le \sigma \le \sigma_2, \tag{2.7}$$

where κ_i, σ_i are positive constants. Then we can prove the following.

THEOREM 2.2. *If* (2.1), (2.6), (2.7) *hold, then there exists a weak solution to* (1.1) *with the boundary conditions* (1.1b) *or* (1.1b′).

Proof. In the case (1.1b) V will be $H_0^1(\Omega)$ and V will be $H^1(\Omega)$ in the case (1.1b′). Choose $w \in L^2(0,T;L^2(\Omega))$; then for almost every $t \in (0,T)$ there exists a unique $\varphi(\cdot,t)$ solution to

$$\nabla \cdot (\sigma(w) \nabla\varphi) = 0 \quad \text{in } \Omega, \qquad \varphi = \varphi_0 \quad \text{on } \Gamma \times (0,T), \tag{2.8}$$

and we have the following.

LEMMA 2.1. $\varphi \in L^\infty(0,T;H^1(\Omega) \cap L^\infty(\Omega))$ *and for almost every t we have*

$$\int_\Omega |\nabla\varphi(x,t)|^2 \, dx \le C(\sigma_1,\sigma_2,\varphi_0), \tag{2.9}$$

where $C(\sigma_1,\sigma_2,\varphi_0)$ denotes a constant depending only on $\sigma_1,\sigma_2,\varphi_0$.

Proof. Assume that we have proved that φ is measurable in t; then from the maximum principle,

$$|\varphi|_\infty \le |\varphi_0|_\infty . \tag{2.10}$$

Moreover, by multiplying the first equation of (2.8) by $\varphi - \varphi_0 \in H_0^1(\Omega)$ we get

$$\int_\Omega \sigma(w) \nabla\varphi \cdot \nabla(\varphi - \varphi_0) = 0;$$

hence

$$\sigma_1 \int_\Omega |\nabla\varphi(x)|^2 \, dx \le \left| \int_\Omega \sigma(w) \nabla\varphi(x) \cdot \nabla\varphi_0 \, dx \right| \le \sigma_2 \int_\Omega |\nabla\varphi(x)| \, |\nabla\varphi_0(x)| \, dx,$$

which gives the result by the Cauchy–Schwarz inequality.

Let us postpone for the time being the proof of the measurability of φ.

Remark that by (1.1c) the right-hand side of (1.1a) can be written as

$$\sigma(u) |\nabla\varphi|^2 = \nabla \cdot (\sigma(u) \varphi \nabla\varphi). \tag{2.11}$$

It is clear then that

$$\langle \nabla \cdot (\sigma(w) \varphi\nabla\varphi), v \rangle = -\int_\Omega \sigma(w) \varphi \nabla\varphi \cdot \nabla v \, dx \quad \forall v \in V$$

defines an element f of $L^2(0,T;V')$. According to Theorem 2.1 there exists a unique u satisfying (2.2)–(2.4) with $\kappa = \kappa(w)$. Let us consider the map

$$w \to u = F(w). \tag{2.12}$$

This map carries $L^2(0,T;L^2(\Omega))$ into itself. Moreover, by (2.5) we have

$$\frac{1}{2} |u(t)|_2^2 + \kappa_1 \int_0^t \|\nabla u(t)\|_2^2 \, dt \le \frac{1}{2} |u(0)|_2^2 - \int_0^t \int_\Omega \sigma(w) \varphi \nabla\varphi \cdot \nabla u \, dx \, dt. \tag{2.13}$$

It follows, by Cauchy–Schwarz and Young's inequalities, that

$$
\begin{aligned}
\frac{1}{2}|u(t)|_2^2+\kappa_1\int_0^t\|\nabla u(t)\|_2^2\,dt &\le \frac{1}{2}|u(0)|_2^2+C\int_0^t\|\nabla\varphi(t)\|_2\|\nabla u(t)\|_2\,dt\\
&\le \frac{1}{2}|u(0)|_2^2+\frac{\kappa_1}{2}\int_0^t\|\nabla u(t)\|_2^2\,dt\\
&\quad+\frac{C^2}{2\kappa_1}\int_0^t\|\nabla\varphi(t)\|_2^2\,dt;
\end{aligned}
\tag{2.14}
$$

hence

$$
|u(t)|_2^2+\int_0^t\|\nabla u(t)\|_2^2\,dt\le C(u_0,T,\kappa_i,\sigma_i,\varphi_0)\,. \tag{2.15}
$$

From (2.3) one easily deduces

$$
|u_t|_{L^2(0,T;V')}\le C'(u_0,T,\kappa_i,\sigma_i,\varphi_0)\,. \tag{2.16}
$$

(Note that f is bounded in $L^2(0,T;V')$ by (2.9), (2.10)). So, provided we take R large enough, $w\to u$ maps the ball B_R of center 0 and radius R in $L^2(0,T;L^2(\Omega))$ into itself. Moreover, since the space

$$
\{u\in L^2(0,T;V)\,|u_t\in L^2(0,T;V')\}
$$

is compactly imbedded in $L^2(0,T;L^2(\Omega))$, this ball will be carried into a relatively compact set by (2.15), (2.16). If we can show that this map is continuous it will be done by the Schauder fixed point theorem. So for that consider a sequence $w_n\in L^2(0,T;L^2(\Omega))$ such that

$$
w_n\to w\quad\text{in } B_R.
$$

Define as in (2.8), $\varphi_n, f_n=\nabla\cdot(\sigma(w_n)\varphi_n\nabla\varphi_n)$, and $u_n=F(w_n)$. We have to show that

$$
u_n\to u=F(w)\quad\text{in } B_R.
$$

For that, by subtracting the equation satisfied by u from the one satisfied by u_n, and taking $v=u_n-u$, we get, after integrating in t,

$$
\begin{aligned}
&\frac{1}{2}|(u_n-u)(t)|_2^2+\kappa_1\int_0^t\|\nabla(u_n-u)(t)\|_2^2\,dt\\
&\le\frac{1}{2}|(u_n-u)(t)|_2^2+\int_0^t\int_\Omega\kappa(w_n)|\nabla(u_n-u)|^2\,dx\,dt\\
&=\int_0^t\int_\Omega(\kappa(w)-\kappa(w_n))\nabla u\cdot\nabla(u_n-u)\,dx\,dt+\int_0^t\langle f_n-f,u_n-u\rangle\,dt\\
&=I_1+I_2.
\end{aligned}
\tag{2.17}
$$

Set

$$I_3 = \frac{\kappa_1}{4} \int_0^t \|\nabla (u_n - u)(t)\|_2^2 \, dt.$$

Then using Young's inequality we get

$$\begin{aligned}
|I_1| &= \left| \int_0^t \int_\Omega (\kappa(w) - \kappa(w_n)) \nabla u \cdot \nabla (u_n - u) \, dx\, dt \right| \\
&\le I_3 + \frac{1}{\kappa_1} \int_0^t \|(\kappa(w) - \kappa(w_n)) \nabla u\|_2^2 \, dt, \\
|I_2| &= \left| \int_0^t \int_\Omega \sigma(w_n) \varphi_n \nabla \varphi_n - \sigma(w) \varphi \nabla \varphi \cdot \nabla (u_n - u) \, dx\, dt \right| \\
&\le I_3 + \frac{1}{\kappa_1} \int_0^t \|\sigma(w_n) \varphi_n \nabla \varphi_n - \sigma(w) \varphi \nabla \varphi\|_2^2 \, dt.
\end{aligned}$$

Thus, taking into account the definition of I_3, we obtain

$$\begin{aligned}
&|(u_n - u)(t)|_2^2 + \int_0^t \|\nabla (u_n - u)(t)\|_2^2 \, dt \\
&\le \frac{1}{\kappa_1} \left[\min\left(\frac{1}{2}, \frac{\kappa_1}{2} \right) \right]^{-1} \left\{ \int_0^T \|(\kappa(w) - \kappa(w_n)) \nabla u\|_2^2 \, dt \right. \\
&\qquad \left. + \int_0^T \|\sigma(w_n) \varphi_n \nabla \varphi_n - \sigma(w) \varphi \nabla \varphi\|_2^2 \, dt \right\}.
\end{aligned} \tag{2.18}$$

Since u_n is in a relatively compact set of B_R it is enough to show that u is the only limit point for u_n. Let u' be such a limit point, i.e.,

$$u' = \lim_{n_k \to \infty} u_{n_k} \quad \text{in } B_R;$$

assuming that we have extracted another sequence of n_k that we still denote by n_k we can assume

$$w_{n_k} \to w \quad \text{a.e. in } \Omega \times (0, T). \tag{2.19}$$

Then, since $|\nabla u|^2 \in L^1(\Omega \times (0,T))$ and by (2.19), $|\kappa(w) - \kappa(w_{n_k})|^2 \to 0$ almost everywhere by the Lebesgue theorem we get

$$\int_0^T \|(\kappa(w) - \kappa(w_{n_k})) \nabla u\|_2^2 \, dt = \int_0^T \int_\Omega |(\kappa(w) - \kappa(w_{n_k}))|^2 |\nabla u|^2 \, dx\, dt \to 0.$$

Next, for $n = n_k$ the second integral in the right-hand side of (2.18) reads

$$\begin{aligned}
&\int_0^T \|\sigma(w_n) \varphi_n \nabla \varphi_n - \sigma(w) \varphi \nabla \varphi\|_2^2 \, dt \\
&\le \int_0^T \|\sigma(w_n) \varphi_n \nabla \varphi_n - \sigma(w_n) \varphi_n \nabla \varphi\|_2^2 \, dt \\
&\quad + \int_0^T \|\sigma(w_n) \varphi_n \nabla \varphi - \sigma(w_n) \varphi \nabla \varphi\|_2^2 \, dt \\
&\quad + \int_0^T \|\sigma(w_n) \varphi \nabla \varphi - \sigma(w) \varphi \nabla \varphi\|_2^2 \, dt \\
&= I + II + III.
\end{aligned}$$

Clearly,

$$I \leq C \int_0^T \int_\Omega |\nabla (\varphi_n - \varphi)|^2 \; dt \; dx,$$

$$II \leq C \int_0^T \int_\Omega |\varphi_n - \varphi|^2 |\nabla \varphi|^2 \; dt \; dx,$$

$$III \leq C \int_0^T \int_\Omega |\sigma (w_n) - \sigma (w)|^2 |\nabla \varphi|^2 \; dt \; dx.$$

By (2.9), (2.19), and from the Lebesgue theorem we can obtain $III \to 0$. Next, φ_n satisfies

$$\nabla \cdot (\sigma (w_n) \nabla \varphi_n) = 0, \qquad \varphi_n = \varphi_0 \quad \text{on } \Gamma.$$

Hence,

$$\int_\Omega \sigma (w_n) \nabla \varphi_n \cdot \nabla (\varphi_n - \varphi) \; dx = \int_\Omega \sigma (w) \nabla \varphi \cdot \nabla (\varphi_n - \varphi) \; dx$$

and

$$\int_\Omega \sigma (w_n) |\nabla (\varphi_n - \varphi)|^2 \; dx = \int_\Omega (\sigma (w) - \sigma (w_n)) \nabla \varphi \cdot \nabla (\varphi_n - \varphi) \; dx,$$

which implies

$$\int_\Omega |\nabla (\varphi_n - \varphi)|^2 \; dx \leq C \int_\Omega |\sigma (w) - \sigma (w_n)|^2 |\nabla \varphi|^2 \; dx. \tag{2.20}$$

Thus,

$$I \leq C \int_0^T \int_\Omega |\nabla (\varphi_n - \varphi)|^2 \; dx \leq C \int_0^T \int_\Omega |\sigma (w) - \sigma (w_n)|^2 |\nabla \varphi|^2 \; dx \to 0$$

as above for *III*. By the Poincaré inequality this implies

$$\int_0^T \int_\Omega |\varphi_n - \varphi|^2 \; dx \to 0,$$

and up to an extracted subsequence we can assume

$$\varphi_n - \varphi \to 0 \quad \text{a.e. on } \Omega \times (0, T);$$

then the Lebesgue convergence theorem gives $II \to 0$ and $u_n \to u = u'$ in $L^2(0, T; L^2(\Omega))$. This completes the proof.

Proof of the measurability of φ. We want to show that φ is measurable in t with values in $H^1(\Omega)$. First remark that if $w \in C([0, T] \times \bar{\Omega})$, then $\varphi \in C([0, T], H^1(\Omega))$. Indeed

$$\nabla \cdot (\sigma (w (t)) \nabla \varphi (t)) = \nabla \cdot (\sigma (w (t')) \nabla \varphi (t')) = 0.$$

Hence,

$$\int_\Omega \sigma(w(t)) |\nabla (\varphi(t) - \varphi(t'))|^2 \; dx = \int_\Omega \sigma(w(t')) - \sigma(w(t)) \nabla \varphi(t') \cdot \nabla (\varphi(t) - \varphi(t')) \; dx$$

and

$$\int_\Omega |\nabla (\varphi(t) - \varphi(t'))|^2 \, dx \le C \int_\Omega |\sigma(w(t')) - \sigma(w(t))|^2 \, |\nabla\varphi(t')|^2 \, dx \to 0$$

when $t \to t'$ by the Lebesgue theorem. Now if $w \in L^2(0,T; L^2(\Omega))$, there exists w_n in $C([0,T] \times \bar{\Omega})$ such that $w_n \to w$ in $L^2(0,T; L^2(\Omega))$, and also almost everywhere on $\Omega \times [0,T]$. From (2.20) we deduce that

$$\int_\Omega |\nabla(\varphi_n - \varphi)|^2 \, dx \to 0,$$

and thus since φ_n is measurable so does φ.

3. Smoothness of weak solutions. Existence of classical solutions. In this section we will assume that (2.7) holds and that

$$|\kappa|_{C^{1+\alpha}(\mathbf{R})}, \; |\sigma|_{C^{1+\alpha}(\mathbf{R})} \le K, \qquad 0 < \alpha < 1, \tag{3.1}$$

where K is some constant. Recall that $C^{1+\alpha}(\mathbf{R})$ denotes the space of C^1 functions with derivatives Hölder continuous of order α, $|\cdot|_{C^{1+\alpha}(\mathbf{R})}$ the usual norm on this space. Ω_t will denote the set $\Omega_t = \Omega \times (0,t)$ and $|\cdot|_{q,r,\Omega_T}$ the usual norm on $L^r(0,T; L^q(\Omega))$ (see [L.S.U.]).

THEOREM 3.1. *Let $w = (u, \varphi)$ be any weak solution of the problem* (1.1) *with the boundary condition* (1.1b) *or* (1.1b′) *such that*

$$|\varphi|_{\infty,\Omega_T} + \|\nabla\varphi\|_{q,r,\Omega_T} \le M_0, \tag{3.2}$$

where (*see* [L.S.U.])

$$0 < \chi < 1, \quad q \in \left[\frac{n}{1-\chi}, +\infty\right], \quad r \in \left[\frac{2}{1-\chi}, +\infty\right], \quad \frac{2}{r} + \frac{n}{q} = 1 - \chi.$$

Then

$$w \in C^{2+\alpha, 1+(\alpha/2)}(\Omega'_T), \qquad \bar{\Omega}'_T \subset \Omega_T$$

and

$$|w|_{C^{2+\alpha,1+(\alpha/2)}(\Omega'_T)} \le C\left(M_0, \text{dist }(\Omega_T \setminus \Omega'_T), \, |u|_{2,\Omega_T}\right). \tag{3.3}$$

If in addition to (3.1), (3.2) *we have*

$$|u_0|_{C^{2+\alpha}(\bar{\Omega})} + |\varphi_0|_{C^{2+\alpha,1+(\alpha/2)}(\Gamma_T)} = H < +\infty, \tag{3.4}$$

$$u_0|_\Gamma = 0 \quad \textit{for } (1.1\text{b}) \quad \textit{or} \quad \left.\frac{\partial u_0}{\partial n}\right|_\Gamma = 0 \quad \textit{for } (1.1\text{b}'), \quad \Gamma_T = \Gamma \times (0,T),$$

then

$$|w|_{C^{2+\alpha,1+(\alpha/2)}(\bar{\Omega}_T)} \le C(M_0, H). \tag{3.5}$$

Proof. The ingredients are well known results of the linear theory of equations of elliptic or parabolic types (see [L.S.U.], [L.U.]). In the formulae below α will be a number between 0 and 1 that may differ from one formula to another.

Step 1. Consider u the solution to the equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (\kappa(u) \nabla u + G), \qquad G = \sigma\varphi\nabla\varphi \in L^q(0,T;L^r(\Omega));$$

then we have

$$|u|_{C^{\alpha,\alpha/2}(\Omega'_T)} \le C_1 \left(M_0, \text{ dist } (\Omega_T \setminus \Omega'_T), |u|_{2,\Omega_T}\right), \tag{3.6}$$

and in the case where (3.4) holds,

$$|u|_{C^{\alpha,\alpha/2}(\bar{\Omega}_T)} \le \bar{C}_1(M_0, H) \tag{3.7}$$

for some $1 > \alpha = \alpha(q,r) > 0$ with $\alpha(q,r) \to 1$ when $(q,r) \to +\infty$.

Step 2. We have $\sigma(u(\cdot,t)) \in C^\alpha(\Omega)$. Then consider φ the solution to the elliptic equation

$$\nabla \cdot (\sigma(u(x,t)) \nabla\varphi) = 0, \qquad \varphi|_\Gamma = \varphi_0.$$

Here t is some parameter and the estimates are not depending on t. We have

$$\sup_{t \le T} |\varphi|_{C^{1+\alpha}(\Omega')} \le C_2(C_1, \text{ dist } (\Omega \setminus \Omega'), M_0), \tag{3.8}$$

respectively, in the case (3.4):

$$\sup_{t \le T} |\varphi|_{C^{1+\alpha}(\bar{\Omega})} \le \bar{C}_2\left(\bar{C}_1, H\right). \tag{3.9}$$

Step 3. From (3.8) and (3.9) we now have

$$G = \sigma\varphi\nabla\varphi \in L^\infty(0,T;L^p(\Omega')) \subset L^p(\Omega'_T), \tag{3.10}$$

respectively,

$$G = \sigma\varphi\nabla\varphi \in L^\infty(0,T;L^p(\Omega)) \subset L^p(\Omega_T) \tag{3.11}$$

for any $p, 1 < p < +\infty$, with

$$|G|_{p,\Omega'_T} \le C_3(C_1, C_2) = C_3\left(M_0, \text{ dist } (\Omega_T \setminus \Omega'_T), |u|_{2,\Omega_T}, p\right) \quad \forall p > 1 \tag{3.12}$$

and in case (3.4):

$$|G|_{p,\Omega_T} \le \bar{C}_3(C_1, C_2, p, H). \tag{3.13}$$

Moreover, (3.6), (3.7) are valid for any $0 < \alpha < 1$ if p is large enough. At the same time we have also

$$|\nabla u|_{C^{\alpha,\alpha/2}(\Omega'_T)} \le C_4(C_3), \tag{3.14}$$

respectively,

$$|\nabla u|_{C^{\alpha,\alpha/2}(\Omega_T)} \le \bar{C}_4\left(\bar{C}_3\right), \tag{3.15}$$

if p is large enough.

Step 4. Consider then the linear elliptic problem

$$\Delta\varphi = -\frac{\sigma'}{\sigma}\nabla u\cdot\nabla\varphi = g \in C^{\alpha}(\Omega), \quad \varphi|_{\Gamma} = \varphi_0.$$

From this equation we deduce

$$\sup_{t\le T}|\varphi|_{C^{2+\alpha}(\Omega')} \le C_5(C_4, M_0), \tag{3.16}$$

respectively,

$$\sup_{t\le T}|\varphi|_{C^{2+\alpha}(\bar\Omega)} \le \bar C_5(\bar C_4, M_0). \tag{3.17}$$

We would like now to show that

$$\nabla\varphi, \varphi_t \in C^{\alpha,\alpha/2}(\Omega_T).$$

Recall that $\nabla\varphi \in C^{\alpha}(\Omega)$ by (3.8) and (3.9). Introduce the function

$$\varphi^{\tau} = \frac{\varphi(x,t+\tau) - \varphi(x,t)}{\tau^{\alpha}} \quad \forall\tau > 0.$$

Then φ^{τ} is a solution to the following elliptic problem:

$$\nabla\cdot(\sigma(u(x,t+\tau))\nabla\varphi^{\tau} + \sigma^{\tau}\nabla\varphi(x,t)) = 0, \qquad \varphi^{\tau}|_{\Gamma} = \varphi_0^{\tau} \tag{3.18}$$

with an obvious notation for σ^{τ}. From (3.6), (3.7), (3.14), and (3.15) we have that

$$\sigma(u(\cdot,t)) \in C^{\alpha}(\Omega), \qquad g(\cdot,t) = \sigma^{\tau}(u(\cdot,t))\nabla\varphi(\cdot,t) \in C^{\alpha}(\Omega),$$

and consequently,

$$\sup_{t\le T}|\nabla\varphi^{\tau}|_{C^{\alpha}(\Omega')} \le C_6(C_1, C_4), \tag{3.19}$$

or in the case of (3.4),

$$\sup_{t\le T}|\nabla\varphi^{\tau}|_{C^{\alpha}(\Omega)} \le \bar C_6(\bar C_1, \bar C_4). \tag{3.20}$$

Hence,

$$\nabla\varphi \in C^{\alpha,\alpha}(\Omega_T) \subset C^{\alpha,\alpha/2}(\Omega_T)$$

and from the equation in u:

$$u_t - \kappa(u)\Delta u = \sigma(u(x,t))|\nabla\varphi|^2 + \kappa'(u)|\nabla u|^2; \tag{3.21}$$

we deduce

$$|u|_{C^{2+\alpha,1+(\alpha/2)}(\Omega'_T)} \le C_7(C_1, C_6), \tag{3.22}$$

respectively,

$$|u|_{C^{2+\alpha,1+(\alpha/2)}(\overline{\Omega_T})} \leq \bar{C}_7\left(\bar{C}_1, \bar{C}_6, H\right). \tag{3.23}$$

We are now able to prove that

$$\varphi_t \in C^{\alpha,\alpha/2}\left(\Omega'_T\right) \qquad \left(\text{respectively}, C^{\alpha,\alpha/2}\left(\overline{\Omega_T}\right)\right).$$

For this remark that

$$\nabla \cdot (\sigma \nabla \varphi_t + \sigma_t \nabla \varphi) = 0, \qquad \varphi_t|_\Gamma = \varphi_{0t}. \tag{3.24}$$

From this equation we derive

$$\nabla \varphi_t \in C^\alpha\left(\Omega'\right) \qquad \left(\text{respectively, } C^\alpha\left(\bar{\Omega}\right)\right) \tag{3.25}$$

and

$$\varphi_t \in C^{2+\alpha}\left(\Omega'\right) \qquad \left(\text{respectively, } C^{2+\alpha}\left(\bar{\Omega}\right)\right)$$

with

$$\sup_{t\leq T} |\varphi_t|_{C^{2+\alpha}(\Omega')} \leq C_8(C_6, C_7) \qquad \left(\text{respectively, } \sup_{t\leq T} |\varphi_t|_{C^{2+\alpha}(\bar{\Omega})} \leq \bar{C}_8(\bar{C}_6, \bar{C}_7)\right). \tag{3.26}$$

Next, we introduce the function

$$\varphi_t^\tau = \frac{\varphi_t\left(x, t+\tau\right) - \varphi_t\left(x,t\right)}{\tau^\alpha}.$$

For φ_t^τ we get the equation

$$\nabla \cdot \left(\sigma\left(x, t+\tau\right) \nabla \varphi_t^\tau + Q\right) = 0, \qquad \varphi_t^\tau|_\Gamma = \varphi_{0t}^\tau \tag{3.27}$$

with

$$Q = \sigma^\tau \nabla \varphi_t^\tau + \sigma_t^\tau \nabla \varphi\left(x, t+\tau\right) + \sigma_t \nabla \varphi^\tau.$$

We have

$$\sup_{t\leq T} |Q|_{C^\alpha(\Omega')} \leq C_9(C_6, C_7, C_8) \qquad \left(\text{respectively, } \sup_{t\leq T} |Q|_{C^\alpha(\bar{\Omega})} \leq \bar{C}_9(\bar{C}_6, \bar{C}_7, \bar{C}_8)\right)$$

from which it follows that

$$|\varphi_t^\tau|_{C^{1+\alpha}(\Omega')} \leq C_{10}(C_9) \quad \forall \tau > 0 \quad \left(\text{respectively, } |\varphi_t^\tau|_{C^{1+\alpha}(\bar{\Omega})} \leq \bar{C}_{10}(\bar{C}_9) \quad \forall \tau > 0\right)$$

or

$$|\nabla \varphi_t^\tau|_{C^{\alpha,\alpha/2}(\Omega'_T)} \leq C_{11}\left(C_{10}\right), \tag{3.28}$$

or in the case where (3.4) holds,

$$|\nabla \varphi_t^\tau|_{C^{\alpha,\alpha/2}(\Omega_T)} \le \bar{C}_{11}\left(\bar{C}_{10}\right). \tag{3.29}$$

This completes the proof.

Remark 3.1. Recall that for any weak solution of the linear elliptic problem

$$\nabla \cdot (\sigma \nabla \varphi) = 0, \qquad \varphi|_\Gamma = \varphi_0 \tag{3.30}$$

we have

$$|\varphi|_\infty \le |\varphi_0|_\infty, \qquad \|\nabla \varphi\|_{p,\Omega} \le C(p)\,\|\nabla \varphi_0\|_{p,\Omega}.$$

Here $p = p(\tau), \tau = \sigma_1/(\sigma_2 - \sigma_1)$ is a given function such that

$$2 < p(\tau), \quad 0 < \tau < \infty, \quad p(\tau) \to +\infty \quad \text{when } \tau \to +\infty, \tag{3.31}$$

which is nondecreasing with τ (recall that $\sigma_1 \le \sigma \le \sigma_2$).

In the two-dimensional case, i.e., when $n = 2$ the assumptions of Theorem 3.2 are fulfilled for $r = \infty, q = p > 2 = n$. Thus, in this case any weak solution to (1.1) is a smooth classical solution in Ω_T (of course, if $\sigma \in C^{1+\alpha}$) and extends smoothly up to the boundary if $u_0 \in C^{2+\alpha}(\bar{\Omega}), \varphi_0 \in C^{2+\alpha,1+(\alpha/2)}(\Gamma)$.

For $n > 2$ the above argument is valid only if σ has a small oscillation in such a way that

$$n < p\left(\frac{\sigma_1}{\sigma_2 - \sigma_1}\right). \tag{3.32}$$

Remark 3.2. To complete Theorem 2.1, the situation regarding existence of a classical solution is the following:

(1) If $n = 2$ for arbitrary smooth σ and any t;

(2) If $n > 2$ for smooth σ with small oscillations and any t;

(3) If $n > 2$ for u_0 with a small oscillation and t small;

(4) If $n > 2$ for t small $(\sigma, u_0$ arbitrary) then (1.1) has a classical solution.

Situations (1) and (2) are clear. To show (3), assume that (2.7), (3.1), and (3.4) hold. Moreover, denote by M a small constant such that

$$-\frac{M}{4} \le u_0(x) \le \frac{M}{4} \tag{3.33}$$

and

$$n < p\left(\frac{\sigma_1^M}{\sigma_2^M - \sigma_1^M}\right) \tag{3.34}$$

with

$$\sigma_1^M = \min_{[-M,+M]} \sigma, \quad \sigma_2^M = \max_{[-M,+M]} \sigma, \quad \sigma_2^M - \sigma_1^M = \operatorname*{osc}_{[-M,+M]} \sigma,$$

$p(\tau)$ being the function of Remark 3.1, osc denoting the oscillation. Define then a function σ^M by

$$\sigma^M(\tau) = \begin{cases} \sigma(\tau) & \text{if } |\tau| \le M/2, \\ \sigma(M) & \text{if } \tau \ge M, \\ \sigma(-M) & \text{if } \tau \le -M, \end{cases} \tag{3.35}$$

and such that

$$\sigma^M \in C^{1+\alpha}, \quad \operatorname*{osc}_{\mathbf{R}} \sigma^M = \sigma_2^M - \sigma_1^M = \operatorname*{osc}_{[-M,+M]} \sigma.$$

Then it is clear that (1.1) corresponding to σ^M has a classical solution (u,φ) for all $t \le T$. Then choose t_0 such that

$$|u(x,t) - u_0(x)| \le \frac{M}{4} \quad \text{or} \quad -\frac{M}{2} \le u(x,t) \le \frac{M}{2} \quad \text{for } t \le t_0.$$

We have for $t \le t_0$,

$$\sigma^M(u(x,t)) = \sigma(u(x,t));$$

hence $u(x,t)$ is a classical solution to (1.1) for $t \le t_0$.

To see (4), introduce the function

$$\sigma^\varepsilon(u,x) = \begin{cases} \sigma(u) & \text{if } |u - u_0(x)| \le \varepsilon/2, \\ \sigma(u_0(x) - \varepsilon) & \text{if } u \le u_0(x) - \varepsilon, \quad x \in \Omega, \\ \sigma(u_0(x) + \varepsilon) & \text{if } u_0(x) + \varepsilon \le u, \end{cases}$$

which is defined for $x \in \Omega, \varepsilon/2 < |u - u_0(x)| < \varepsilon$ so that

$$\sigma^\varepsilon(u,x) \in C^{1+\alpha}(\mathbf{R} \times \Omega), \qquad \operatorname*{osc}_{\mathbf{R}\times\Omega} \sigma^\varepsilon(u,x) = \operatorname*{osc}_{[|u-u_0(x)|\le\varepsilon, x\in\Omega]} \sigma^\varepsilon(u,x).$$

Clearly, $\sigma^\varepsilon(u,x) \to \sigma(u_0(x))$ when $\varepsilon \to 0$. We select ε small enough such that if

$$\lambda^\varepsilon = \frac{\sigma^\varepsilon(u,x)}{\sigma(u_0(x))}, \qquad \tau = \min_{\mathbf{R}\times\Omega} \frac{\lambda^\varepsilon}{(\max \lambda^\varepsilon - \min \lambda^\varepsilon)},$$

we have

$$n < p(\tau).$$

Consider now the problem (1.1), where $\sigma(u)$ is replaced by $\sigma^\varepsilon(u,x)$. The equation for φ reads

$$\nabla \cdot (\sigma^\varepsilon \nabla \varphi) = 0.$$

By Theorem 2.1, there exists a weak solution to problem (1.1) corresponding to $\sigma = \sigma^\varepsilon(u,x)$. Let us show that this solution is in fact classical. Introduce $v = \sigma(u_0(x))\varphi(x,t)$. Then, v satisfy

$$\nabla \cdot [\lambda^\varepsilon (\nabla v - v \nabla \ \ln \ \sigma(u_0(x)))] = 0.$$

Note that $\lambda^\varepsilon \to 1$ when $\varepsilon \to 0$. According to the fact that $n < p(\tau)$ and (2) we have

$$\nabla v \in L^\infty(0,T;L^p(\Omega)), \qquad p > n,$$

and thus

$$\nabla \varphi = \frac{1}{\sigma(u_0(x))} (\nabla v - \varphi \nabla \sigma(u_0(x))) \in L^\infty(0,T;L^p(\Omega)).$$

Then, by Theorem 3.1, we have

$$u \in C^{2+\alpha,1+(\alpha/2)}\left(\bar{\Omega}_T\right).$$

Hence

$$|u(x,t) - u_0(x,t)| \leq C(\varepsilon)\, t.$$

Selecting t such that $C(\varepsilon)t < \varepsilon/2$ we have $\sigma^\varepsilon(u,x) = \sigma(u)$, and thus the existence of a classical solution for small t is established.

Remark 3.3. So we have existence of a classical solution to (1.1) for small t. To extend this solution for all $t \leq T$ we need estimates for $t \leq T$. According to Theorem 3.1 the estimate (see (3.2))

$$\|\nabla\varphi\|_{q,r,\Omega_T} \leq M, \qquad \frac{2}{r} + \frac{n}{q} = 1 - \chi$$

is enough. We are now going to establish this estimate for

$$r = q = \frac{2+n}{1-\chi} > 2 + n.$$

Indeed we have the following.

THEOREM 3.2. *Let (u, φ) be a classical solution to the problem* (1.1) *and assume that*

$$0 < \sigma_1 \leq \sigma \leq \sigma_2 < +\infty, \qquad |\sigma'| \leq K \tag{3.36}$$

$$\sup_{0\leq t\leq T}\left(|\varphi_0|_{C^\alpha(\bar{\Omega})}; \|\nabla\varphi_0\|_{p,\Omega}\right) \leq M, \qquad p > 2; \tag{3.37}$$

then for $2s + 2 > n$ and any $T < +\infty$,

$$\|\nabla u\|_{2s+2,\Omega_T} + \|\nabla\varphi\|_{2s+2,\Omega_T} \leq C\left\{\|u_0\|^{(1)}_{2s+2,\Omega_T} + \|\varphi_0\|^{(1)}_{2s+2,\Omega_T}\right\}, \tag{3.38}$$

where $C = c(s, n, T, \Omega, p, K, \sigma_i, M)$, and

$$\|f\|^{(1)}_{k,\Omega_T} = |f|_{k,\Omega_T} + \|\nabla f\|_{k,\Omega_T}, \qquad |f|_{k,\Omega_T} = |f|_{k,k,\Omega_T}.$$

Proof. The proof goes through several steps. The scheme is the following.

Step 1. Considering t as a parameter we derive local estimates inside Ω for any $t \leq T$ for the solution to the problem

$$\nabla \cdot (\sigma(u)\nabla\varphi) = 0, \qquad \varphi|_\Gamma = \varphi_0. \tag{3.39}$$

Step 2. We derive local estimates for the solution u to the problem

$$u_t - \nabla \cdot (\kappa(u)\nabla u) = \nabla \cdot (\sigma(u)\varphi\nabla\varphi) = \sigma(u)|\nabla\varphi|^2, \tag{3.40}$$

where $(\sigma(u)\varphi\nabla\varphi) = \sigma(u)|\nabla\varphi|^2$ is considered as a given function of x and t.

Step 3. We deduce global estimates for (u, φ).

Let us first go through Step 1.

Step 1. Let us denote by $\xi_k, k = 1, \ldots, m$ smooth functions such that

$$\sum_1^m \xi_k^{2s+2} = 1, \quad x \in \bar{\Omega}, \quad 0 \leq \xi_k(x), \quad |\xi, \nabla\xi, \nabla^2\xi| \leq 1,$$

the diameter of their support being smaller than some small number that we will choose later on.

First we have the following.

LEMMA 3.1. *Suppose that φ is a classical solution to*

$$\nabla \cdot (\sigma(u) \nabla\varphi) = 0, \quad x \in \Omega, \quad \varphi|_\Gamma = \varphi_0$$

and that $\xi_k(x)$ is a smooth function such that $(\varphi - \varphi_0)\xi_k(x)$ vanishes outside of the domain $\Omega_k \subset \bar{\Omega}$. Then, for any $2s + 2 > n$ and $t \leq T$,

(3.41)

$$\int_\Omega |\nabla\varphi|^{2s+2} \cdot \xi_k^{2s+2}\, dx \leq C\left\{\int_\Omega \left(\delta^{2s+2}|\nabla u|^{2s+2} + |\nabla\varphi_0|^{2s+2}\right) \cdot \xi_k^{2s+2}\, dx + I_1(k)\right\}.$$

In the above inequality $C = C(s, n, k), \delta = \operatorname{osc}_{\Omega_k}(\varphi - \varphi_0)$,

(3.42)

$$I_1(k) = \int_{\Omega_k} \left[\delta^{2s+2}|\nabla\xi_k|^{2s+2} + \delta^{s+1}|\varphi - \varphi_0|^{s+1}\left(|\nabla\xi_k|^{2s+2} + \xi_k^{s+1}|\nabla^2\xi_k|^{s+1}\right)\right] dx,$$

φ_0 being the function such that

$$\Delta\varphi_0 = 0, \qquad \varphi_0|_\Gamma = \varphi_0.$$

In particular, if

$$\xi_k \equiv 1,$$

then

$$\int_\Omega |\nabla\varphi|^{2s+2}\, dx \leq C\left\{\int_\Omega |\nabla\varphi_0|^{2s+2} + |\nabla u|^{2s+2}\right\} dx, \tag{3.43}$$

where $C = C(s, n, k, |\varphi_0|_\infty)$.

Proof. The proof is similar to the one in [A.K.M., p. 254]. It is based on [L.S.U., p. 94, form. 5.8]:

(3.44)

$$\begin{aligned}\int_\Omega &|\nabla(\varphi - \varphi_0)|^{2s+2} \cdot \xi_k^{2s+2}\, dx \\ &\leq 16\left(\operatorname*{osc}_{\Omega_k}(\varphi - \varphi_0)\right)^2 \left\{\int_\Omega c^2|\nabla(\varphi - \varphi_0)|^{2s-2}|\nabla^2(\varphi - \varphi_0)|^2 \xi_k^{2s+2}\right. \\ &\qquad\left. + |\nabla(\varphi - \varphi_0)|^{2s} \cdot |\nabla\xi_k|^2 \xi_k^{2s}(s+1)^2\, dx\right\}, \quad c^2 = n^2 + s^2.\end{aligned}$$

From this inequality by Young's inequality and local estimates of $\nabla^2\varphi$ in terms of $\Delta\varphi$ we deduce

$$\begin{aligned}\int_\Omega |\nabla\varphi|^{2s+2}\cdot\xi_k^{2s+2}\,dx \le C\Big[\delta^{s-1}\int_\Omega |\Delta\varphi|^{s+1}\cdot\xi_k^{2s+2}\,dx \\ + \int_\Omega |\nabla\varphi_0|^{2s+2}\cdot\xi_k^{2s+2}\,dx + I_1(k)\Big].\end{aligned}\tag{3.45}$$

We now use the elliptic equation

$$\Delta\varphi = -\frac{\sigma'}{\sigma}\,\nabla u\cdot\nabla\varphi$$

to deduce

$$\int_{\Omega_k} |\Delta\varphi|^{2s+2}\cdot\xi_k^{2s+2}\,dx \le C'\int_{\Omega_k}\left[\frac{1}{2\varepsilon}|\nabla\varphi|^{2s+2} + \frac{\varepsilon}{2}|\nabla u|^{2s+2}\right]\cdot\xi_k^{2s+2}\,dx \tag{3.46}$$

for some constant C'. Substituting (3.46) in the right-hand side of (3.45) with $\varepsilon = CC'\delta^{s+1}$ we obtain (3.41).

Step 2. Next we have the following.

LEMMA 3.2. *Let $u(x,t)$ be a classical solution to*

$$u_t = \nabla\cdot(\kappa(u)\nabla u + \sigma\varphi\nabla\varphi), \quad u(0)=u_0, \quad u|_\Gamma = 0, \quad \text{or} \quad \left.\frac{\partial u}{\partial n}\right|_\Gamma = 0 \tag{3.47}$$

and ξ_k as in the preceding lemma. Then, for any $2s+2>n$ and $t\le T$,

$$\int_0^t\int_\Omega |\nabla u|^{2s+2}\cdot\xi_k^{2s+2}\,dx\,d\tau \le C\left[\int_0^t\int_\Omega |\nabla\varphi|^{2s+2}\cdot\xi_k^{2s+2}\,dx\,d\tau + I_2(k,t)\right], \tag{3.48}$$

where $C = C(n,s,\Omega,T,\sigma_1,|\varphi|_\infty)$,

$$\begin{aligned}I_2(k,t) = \left(\|u_0(x)\,\xi_k\|^{(1)}_{2s+2,\Omega_k}\right)^{2s+2} + \int_0^t \|u\nabla\xi_k\|^{2s+2}_{2s+2,\Omega}\,d\tau \\ + \int_0^t \|\nabla\theta_k\|^{2s+2}_{2s+2,\Omega_k}\,d\tau.\end{aligned}\tag{3.49}$$

$|f|^{(1)}_{k,\Omega} = |f|_{k,\Omega} + \|\nabla f\|_{k,\Omega}$, $\theta_k(x,t)$ is the solution to the problem

$$\Delta\theta_k = -\nabla\xi_k\cdot(\kappa(u)\nabla u + \sigma\varphi\nabla\varphi), \quad x\in\Omega_k, \quad \theta_k|_{\Gamma_k} = 0. \tag{3.50}$$

In particular, if $\xi_k\equiv 1$, then

$$\int_0^t\int_\Omega |\nabla u|^{2s+2}\,dx\,d\tau \le C\left[\int_0^t\int_\Omega |\nabla\varphi|^{2s+2}\,dx\,d\tau + \left(\|u_0\|^{(1)}_{2s+2,\Omega}\right)^{2s+2}\right]. \tag{3.51}$$

Proof. Introduce

$$u^k(x,t) = u(x,t)\,\xi_k(x).$$

Then

$$\frac{\partial u^k}{\partial t} = \nabla \cdot \left(\kappa(u)\nabla u^k + \sigma\varphi\nabla\varphi\xi_k + G_k\right), \quad x \in \Omega_k, \quad u^k|_{\Gamma k} = 0, \tag{3.52}$$
$$u^k(0) = u_0\xi_k, \quad x \in \Omega_k, \quad G_k = -\left(\kappa(u)u\nabla\xi_k + \nabla\theta_k\right).$$

We then deduce (see [L.S.U., Thm. 8.1, 8, Thm. 10.1, 10, Chap. III] and [A.K.M., Thm. 1, p. 230])

$$\begin{aligned}\|\nabla u^k\|_{q,\Omega_t}^q &\le C\left(\|G_k\|_{q,\Omega_t}^q + \|\nabla\varphi\xi_k\|_{q,\Omega_t}^q + \|u_0\xi_k\|_{q,\Omega}^{(1)}\right)\\ &\le C\left[\|\nabla\varphi\xi_k\|_{q,\Omega_t}^q + I_2(k,t)\right].\end{aligned}$$

Hence

$$\|\nabla u\xi_k\|_{q,\Omega_t}^q \le C\left(\|\nabla\varphi\xi_k\|_{q,\Omega_t}^q + I_2\right)$$

(this for any $1 < q < \infty, t \le T, C = C(\Omega_k, T, n, q)$). When $q = 2s+2$ we get (3.48), and the proof of Lemma 3.2 is complete.

Step 3. Substituting (3.41) into the right-hand side of (3.48) and choosing the domain Ω_k small enough in such a way that

$$C^2\delta^{2s+2} \le \tfrac{1}{2},$$

we obtain

$$\int_0^t \int_\Omega |\nabla u|^{2s+2}\xi_k^{2s+2}\,dx\,d\tau \le C\left[I_1(k) + I_2(k\,t)\right]. \tag{3.53}$$

From (3.42) we have, $(2s+2 > n)$,

$$|I_1(k)| \le C\|\varphi - \varphi_0\|_{\infty,\Omega}^{2s+2} \le 2C\|\varphi_0\|_{\infty,\Omega}^{2s+2} \le \tilde{C}\left(\|\varphi_0\|_{\infty,\Omega}^{(1)}\right)^{2s+2}. \tag{3.54}$$

From (3.49) we also get

$$|I_2(k,t)| \le C\left[\left(\|u_0\|_{2s+2,\Omega}^{(1)}\right)^{2s+2} + \|u\|_{2s+2,\Omega_t}^{2s+2} + \int_0^t \|\nabla\theta_k\|_{2s+2,\Omega_k}^{2s+2}\,d\tau\right]. \tag{3.55}$$

For the solution to the problem (3.50) we have the following representation formula:

$$\theta_k = P\left(\left(\kappa(u)\nabla u + \sigma\varphi\nabla\varphi\right)\nabla\xi_k\,|x\right), \qquad \left(\kappa(u)\nabla u = \nabla\left(\int_0^u \kappa(s)\,ds\right)\right),$$

where

$$P(g|\,x) = \int_{\Omega_k} I(x-y)\,g(y)\,dy,$$

I being Green's function. Thus

$$\begin{aligned}\nabla\theta_k = &-\int_{\Omega_k}\nabla I\cdot\Delta\xi_k\kappa(u)u\,dy - \int_{\Omega_k}\nabla^2 I\cdot\nabla\xi_k\int_0^{u(x,t)}\kappa(s)\,ds\,dy\\ &+\int_{\Omega_k}\nabla I\cdot\nabla\xi_k\sigma\varphi\nabla\varphi\,dy.\end{aligned} \tag{3.56}$$

By the properties of the operator P (see [L.U.]) and (3.56) we deduce

(3.57)

$$\begin{aligned}\|\nabla\theta_k\|_{2s+2,\Omega_k} &\le C\left[\|u\|_{(2s+2)n/(2s+2+n),\Omega} + \|u\|_{2s+2,\Omega} + \|\nabla\varphi\|_{(2s+2)n/(2s+2+n),\Omega}\right]\\ &\le C\left[\|u\|_{2s+2,\Omega} + \|\nabla\varphi\|_{(2s+2)n/(2s+2+n),\Omega}\right].\end{aligned}$$

From (3.53), (3.54), (3.55), (3.57) we have

(3.58)

$$\begin{aligned}\int_0^t \|\nabla u\|_{2s+2,\Omega}^{2s+2}\,dt &\le C\left[\left(\|u_0\|_{2s+2,\Omega}^{(1)}\right)^{2s+2} + |u|_{2s+2,\Omega_t}^{2s+2} + \int_0^t \left(\|\varphi_0\|_{2s+2,\Omega}^{(1)}\right)^{2s+2} dt\right.\\ &\qquad\left. + \int_0^t \left(\|\nabla\varphi\|_{(2s+2)n/(2s+2+n),\Omega}\right)^{2s+2} dt\right]\\ &\equiv C\left[H(u,\varphi) + |u|_{2s+2,\Omega_t}^{2s+2} + \int_0^t \left(\|\nabla\varphi\|_{(2s+2)n/(2s+2+n),\Omega}\right)^{2s+2} dt\right]\\ &\equiv Q.\end{aligned}$$

From (3.43) we also get, for φ,

$$\|\nabla\varphi\|_{2s+2,\Omega_t}^{2s+2} + \|\nabla u\|_{2s+2,\Omega_t}^{2s+2} \le 2Q. \tag{3.59}$$

Moreover, we have

$$\begin{aligned}\|u\|_{2s+2,\Omega_t}^{2s+2} &\le \varepsilon\|\nabla u\|_{2s+2,\Omega_t}^{2s+2} + C_\varepsilon \int_0^t |u|_{2,\Omega}^{2s+2}\,dt\\ &\le \varepsilon\|\nabla u\|_{2s+2,\Omega_t}^{2s+2} + \tilde{C}_\varepsilon\left(\int_0^t |u_0|_{2,\Omega}^2 + \int_0^\tau \|\nabla\varphi\|_{2,\Omega}^2\,d\tau\right)^{(2s+2)/2}\\ &\le \varepsilon\|\nabla u\|_{2s+2,\Omega_t}^{2s+2} + \tilde{C}'_\varepsilon H\end{aligned}\tag{3.60}$$

and

$$\begin{aligned}\int_0^t \|\nabla\varphi\|_{q,\Omega}^{2s+2}\,dt &\le \int_0^t \|\nabla\varphi\|_{p,\Omega}^{(2s+2)/q}\,\|\nabla\varphi\|_{(q-1)p/(p-1),\Omega}^{(2s+2)(q-1)/q}\,dt\\ &\le C\int_0^t \|\nabla\varphi_0\|_{p,\Omega}^{(2s+2)/q}\,\|\nabla\varphi\|_{2s+2,\Omega}^{(2s+2)(q-1)/q}\,dt\\ &\le \varepsilon\|\nabla\varphi\|_{2s+2,\Omega_t}^{2s+2} + C_\varepsilon H\end{aligned}\tag{3.61}$$

($q = (2s+2)n/(n+2s+2), p > 2$). Combining (3.59)–(3.61) we obtain (3.58), and the Theorem (3.2) is proved.

Remark 3.4. The estimate (3.38) allows us to prove the existence of a solution to (1.1) in the space of (u,φ) such that

$$(\nabla u, \nabla\varphi) \in L^{2s+2}(0,T;L^{2s+2}(\Omega)), \qquad (u_t, \nabla^2 u, \nabla^2\varphi) \in L^{s+1}(0,T;L^{s+1}(\Omega)).$$

4. Dependence on the data and uniqueness results. In this section we will assume that (2.6) and (2.7) hold and that κ, σ are Lipschitz continuous, i.e., that for some constant K,

$$|\kappa(u_1) - \kappa(u_2)|, \qquad |\sigma(u_1) - \sigma(u_2)| \le K|u_1 - u_2| \quad \forall u_1, u_2 \in \mathbf{R}. \tag{4.1}$$

Then we have the following.

THEOREM 4.1. *Let $(u_i, \varphi_i), i = 1, 2,$ two weak solutions to (1.1) with the boundary conditions (1.1b) or (1.1b$'$) corresponding to the data $(u_0^i, \varphi_0^i, \kappa^i, \sigma^i)$. Assume that (2.1), (2.6), (2.7), and (4.1) hold for $(u_0^i, \varphi_0^i, \kappa^i, \sigma^i), i = 1, 2,$ and also that*

$$\nabla u_i, \nabla \varphi_i \in L^{2q/(q-n)}(0, T; L^q(\Omega)), \qquad q > n \vee 2, \quad i = 1, 2, \tag{4.2}$$

where $n \vee 2$ denotes the maximum of 2 and n. Then we have

$$\begin{aligned} &|w(t)|_2^2 + \int_0^t \|\nabla w(\tau)\|_2^2 \, d\tau + \int_0^t \|\nabla \varphi(\tau)\|_2^2 \, d\tau \\ &\le C\left(|w_0|_2^2 + |\kappa|_\infty^2 + |\sigma|_\infty^2 + \int_0^t \|\nabla \varphi_0\|_2^2 \, d\tau\right) \quad \forall t \le T, \end{aligned} \tag{4.3}$$

where

$$w = u_1 - u_2, \quad \varphi = \varphi_1 - \varphi_2, \quad w_0 = u_0^1 - u_0^2, \quad \varphi_0 = \varphi_0^1 - \varphi_0^2,$$
$$\kappa = \kappa^1 - \kappa^2, \qquad \sigma = \sigma^1 - \sigma^2,$$
$$|\kappa|_\infty = \sup_{\tau \in \mathbf{R}} |\kappa(\tau)|, \; |\sigma|_\infty = \sup_{\tau \in \mathbf{R}} |\sigma(\tau)|, \; C = C\left(T, \|\nabla u_i\|_{q,2q/(q-n)}, \|\nabla \varphi_i\|_{q,2q/(q-n)}\right).$$

Proof. Subtracting the equation satisfied by u_2 from the one satisfied by u_1 we obtain

$$\begin{aligned} w_t &= \nabla \cdot (\kappa^1(u_1)\nabla u_1 - \kappa^2(u_2)\nabla u_2) + \sigma^1(u_1)|\nabla\varphi_1|^2 - \sigma^2(u_2)|\nabla\varphi_2|^2 \\ &= \nabla \cdot (\kappa^1(u_1)\nabla w) + \nabla \cdot (\kappa^1(u_1) - \kappa^1(u_2)\nabla u_2) \\ &\quad + \nabla \cdot (\kappa^1(u_2) - \kappa^2(u_2)\nabla u_2) + (\sigma^1(u_1) - \sigma^1(u_2))|\nabla\varphi_1|^2 \\ &\quad + \sigma^1(u_2)\nabla\varphi \cdot (\nabla\varphi_1 + \nabla\varphi_2) + (\sigma^1(u_2) - \sigma^2(u_2))|\nabla\varphi_2|^2. \end{aligned}$$

If we multiply by w and integrate over Ω we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|w|_2^2 + \kappa_1 \int_\Omega |\nabla w|^2 \, dx &\le \frac{1}{2}\frac{d}{dt}|w|_2^2 + \int_\Omega \kappa^1(u_1)|\nabla w|^2 \, dx \\ &= \int_\Omega \kappa^1(u_2) - \kappa^1(u_1)\nabla u_2 \cdot \nabla w \, dx \\ &\quad + \int_\Omega \kappa^2(u_2) - \kappa^1(u_2)\nabla u_2 \cdot \nabla w \, dx \\ &\quad + \int_\Omega (\sigma^1(u_1) - \sigma^1(u_2))|\nabla\varphi_1|^2 w \, dx \\ &\quad + \int_\Omega \sigma^1(u_2)\nabla\varphi \cdot (\nabla\varphi_1 + \nabla\varphi_2) w \, dx \\ &\quad + \int_\Omega (\sigma^1(u_2) - \sigma^2(u_2))|\nabla\varphi_2|^2 w \, dx. \end{aligned} \tag{4.4}$$

Using (4.1) and Hölder's and Young's inequalities we easily see that

(4.5)

$$\begin{aligned}
\left|\int_\Omega \kappa^1(u_2) - \kappa^1(u_1)\nabla u_2 \cdot \nabla w\, dx\right| &\le K\int_\Omega |\nabla u_2\,\|\nabla w\|\,w|\;dx \\
&\le K\,|\nabla u_2|_q\,\|\nabla w\|_2\,|w|_{2q/(q-2)} \\
&\le \varepsilon\,\|\nabla w\|_2^2 + C_\varepsilon\,\|\nabla u_2\|_q^2\,|w|_{2q/(q-2)}^2, \\
\left|\int_\Omega \kappa^2(u_2) - \kappa^1(u_2)\nabla u_2 \cdot \nabla w\, dx\right| &\le |\kappa|_\infty\,\|\nabla u_2\|_2\,\|\nabla w\|_2 \\
&\le \varepsilon\,\|\nabla w\|_2^2 + C_\varepsilon\,|\kappa|_\infty^2\,\|\nabla u_2\|_q^2, \\
\left|\int_\Omega (\sigma^1(u_1) - \sigma^1(u_2))\,|\nabla\varphi_1|^2\,w\, dx\right| &\le K\int_\Omega |\nabla\varphi_1|^2\,|w|^2\;dx \\
&\le K\,\|\nabla\varphi_1\|_q^2\,|w|_{2q/(q-2)}^2, \\
\left|\int_\Omega \sigma^1(u_2)\nabla\varphi\cdot(\nabla\varphi_1+\nabla\varphi_2)\,w\, dx\right| &\le \sigma_2\,\|\nabla\varphi\|_2\,\|\nabla(\varphi_1+\varphi_2)\|_q\,|w|_{2q/(q-2)} = I, \\
\left|\int_\Omega (\sigma^1(u_2) - \sigma^2(u_2))\,|\nabla\varphi_2|^2\,w\, dx\right| &\le \|\nabla\varphi_2\|_q^2\,|w|_{2q/(q-2)}\,|\sigma|_\infty \\
&\le C\,\|\nabla\varphi_2\|_q^2\left(|w|_{2q/(q-2)}^2 + |\sigma|_\infty^2\right).
\end{aligned}$$

To estimate I, we need to estimate φ. So, we use the equation satisfied by φ_1 and φ_2 to get

$$\begin{aligned}
-\nabla\cdot(\sigma^1(u_1)\nabla\varphi) = &-\nabla\cdot(\sigma^1(u_2) - \sigma^1(u_1)\nabla\varphi_2) \\
&-\nabla\cdot(\sigma^2(u_2) - \sigma^1(u_2)\nabla\varphi_2).
\end{aligned}\tag{4.6}$$

Multiplying this equation by $\varphi - \varphi_0$ and integrating over Ω leads to

(4.7)

$$\begin{aligned}
\int_\Omega \sigma^1(u_1)\,|\nabla\varphi|^2\;dx = &\int_\Omega \sigma^1(u_2) - \sigma^1(u_1)\nabla\varphi_2\cdot\nabla\varphi\;dx \\
&+\int_\Omega \sigma^2(u_2) - \sigma^1(u_2)\nabla\varphi_2\cdot\nabla\varphi\;dx + \int_\Omega \sigma^1(u_1)\nabla\varphi\cdot\nabla\varphi_0\;dx \\
&-\int_\Omega \sigma^1(u_2) - \sigma^1(u_1)\nabla\varphi_2\cdot\nabla\varphi_0\;dx \\
&-\int_\Omega \sigma^2(u_2) - \sigma^1(u_2)\nabla\varphi_2\cdot\nabla\varphi_0\;dx.
\end{aligned}$$

From this equality it follows that

$$\begin{aligned}
\|\nabla\varphi\|_2^2 \le C\Big\{&\|\nabla\varphi\|_2\left(\|\nabla\varphi_2\|_q\,|w|_{2q/(q-2)} + |\sigma|_\infty\,\|\nabla\varphi_2\|_2 + \|\nabla\varphi_0\|_2\right) \\
&+\|\nabla\varphi_2\|_q\,\|\nabla\varphi_0\|_2\,|w|_{2q/(q-2)} + \|\nabla\varphi_2\|_2\,\|\nabla\varphi_0\|_2\,|\sigma|_\infty\Big\},
\end{aligned}$$

and by Young's inequality,

$$\|\nabla\varphi\|_2^2 \le \varepsilon\,\|\nabla\varphi\|_2^2 + C_\varepsilon\left(\|\nabla\varphi_2\|_q^2\,|w|_{2q/(q-2)}^2 + |\sigma|_\infty^2\,\|\nabla\varphi_2\|_2^2 + \|\nabla\varphi_0\|_2^2\right).\tag{4.8}$$

We thus obtain

$$\begin{aligned}\|\nabla\varphi\|_2 &\le C\left(\|\nabla\varphi_2\|_q |w|_{2q/(q-2)} + |\sigma|_\infty \|\nabla\varphi_2\|_2 + \|\nabla\varphi_0\|_2\right)\\ &\le C\left(\|\nabla\varphi_2\|_q |w|_{2q/(q-2)} + |\sigma|_\infty \|\nabla\varphi_2\|_q + \|\nabla\varphi_0\|_2\right),\end{aligned}$$

and so

$$\begin{aligned}I &\le C\left[\left\{\|\nabla\varphi_1\|_q^2 + \|\nabla\varphi_2\|_q^2\right\}|w|_{2q/(q-2)}^2\right.\\ &\qquad \left.+ \|\nabla(\varphi_1+\varphi_2)\|_q |w|_{2q/(q-2)}\left\{|\sigma|_\infty \|\nabla\varphi_2\|_q + \|\nabla\varphi_0\|_2\right\}\right]\\ &\le C\left[\left\{\|\nabla\varphi_1\|_q^2 + \|\nabla\varphi_2\|_q^2\right\}|w|_{2q/(q-2)}^2 + |\sigma|_\infty^2 \|\nabla\varphi_2\|_q^2 + \|\nabla\varphi_0\|_2^2\right].\end{aligned} \tag{4.9}$$

Collecting (4.4), (4.5), and (4.9) and choosing $\varepsilon = \kappa_1/6$ in (4.5), we get

(4.10)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|_2^2 + \frac{2\kappa_1}{3}\|\nabla w\|_2^2 &\le C\left\{\|\nabla u_2\|_q^2 + \|\nabla\varphi_1\|_q^2 + \|\nabla\varphi_2\|_q^2\right\}|w|_{2q/(q-2)}^2\\ &\quad + C\left\{|\kappa|_\infty^2 \|\nabla u_2\|_q^2 + |\sigma|_\infty^2 \|\nabla\varphi_2\|_q^2 + \|\nabla\varphi_0\|_2^2\right\}.\end{aligned}$$

From the Gagliardo-Nirenberg interpolation inequality we have for some constant C,

$$|w|_{2q/(q-2)} \le C|w|_2^{1-(n/q)}\left(|w|_2^2 + \|\nabla w\|_2^2\right)^{n/2q} \quad \forall w \in H^1(\Omega). \tag{4.11}$$

Hence (4.10) becomes

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|_2^2 + \frac{2\kappa_1}{3}\|\nabla w\|_2^2 &\le C\left\{\|\nabla u_2\|_q^2 + \|\nabla\varphi_1\|_q^2 + \|\nabla\varphi_2\|_q^2\right\}|w|_2^{2(1-(n/q))}\\ &\quad \cdot\left(|w|_2^2 + \|\nabla w\|_2^2\right)^{n/q}\\ &\quad + C\left\{|\kappa|_\infty^2 \|\nabla u_2\|_q^2 + |\sigma|_\infty^2 \|\nabla\varphi_2\|_q^2 + \|\nabla\varphi_0\|_2^2\right\}.\end{aligned} \tag{4.12}$$

Hence by applying the Young inequality

$$ab \le \varepsilon a^{q/n} + C_\varepsilon b^{q/(q-n)},$$

it follows that for any $\varepsilon > 0$,

(4.13)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|w|_2^2 + \frac{2\kappa_1}{3}\|\nabla w\|_2^2 &\le 3\varepsilon\left(|w|_2^2 + \|\nabla w\|_2^2\right)\\ &\quad + C_\varepsilon\left(\|\nabla u_2\|_q^{2q/(q-n)} + \|\nabla\varphi_1\|_q^{2q/(q-n)} + \|\nabla\varphi_2\|_q^{2q/(q-n)}\right)|w|_2^2\\ &\quad + C\left\{|\kappa|_\infty^2 \|\nabla u_2\|_q^2 + |\sigma|_\infty^2 \|\nabla\varphi_2\|_q^2 + \|\nabla\varphi_0\|_2^2\right\},\end{aligned}$$

where C_ε is some constant depending on ε. Hence, by choosing $3\varepsilon = \kappa_1/6$,

(4.14)

$$\frac{d}{dt}|w|_2^2 + \kappa_1 \|\nabla w\|_2^2 \le C\left[\left(1 + \|\nabla u_2\|_q^{2q/(q-n)} + \|\nabla \varphi_1\|_q^{2q/(q-n)} + \|\nabla \varphi_2\|_q^{2q/(q-n)}\right)|w|_2^2 + |\kappa|_\infty^2 \|\nabla u_2\|_q^2 + |\sigma|_\infty^2 \|\nabla \varphi_2\|_q^2 + \|\nabla \varphi_0\|_2^2\right].$$

If we set

$$[w(t)] = |w|_2^2 + \kappa_1 \int_0^t \|\nabla w\|_2^2 \, d\tau, \tag{4.15}$$

$$H = C\left(1 + \|\nabla u_2\|_q^{2q/(q-n)} + \|\nabla \varphi_1\|_q^{2q/(q-n)} + \|\nabla \varphi_2\|_q^{2q/(q-n)}\right) \in L^1(0,T) \tag{4.16}$$

(see (4.2)), then (4.14) also reads

$$\frac{d}{dt}[w] - H[w] \le |\kappa|_\infty^2 \|\nabla u_2\|_q^2 + |\sigma|_\infty^2 \|\nabla \varphi_2\|_q^2 + \|\nabla \varphi_0\|_2^2$$

or

$$\frac{d}{dt}\left(e^{-\int_0^t H(s)ds}[w]\right) \le e^{-\int_0^t H(s)ds}\left\{|\kappa|_\infty^2 \|\nabla u_2\|_q^2 + |\sigma|_\infty^2 \|\nabla \varphi_2\|_q^2 + \|\nabla \varphi_0\|_2^2\right\}.$$

Hence, integrating between o and t,

$$\begin{aligned}[w] &\le [w(0)] + e^{\int_0^t H(s)ds}\left(\int_0^t e^{-\int_0^t H(s)ds}\left\{|\kappa|_\infty^2 \|\nabla u_2\|_q^2 + |\sigma|_\infty^2 \|\nabla \varphi_2\|_q^2 + \|\nabla \varphi_0\|_2^2\right\} d\tau\right) \\ &= |w_0|_2^2 + C(T)\left(|\kappa|_\infty^2 \left\{\int_0^t \|\nabla u_2\|_q^{2q/(q-n)} \, d\tau\right\}^{(q-n)/q}\right. \\ &\qquad \left. + |\sigma|_\infty^2 \left\{\int_0^t \|\nabla \varphi_2\|_q^{2q/(q-n)} \, d\tau\right\}^{(q-n)/q} + \int_0^t \|\nabla \varphi_0\|_2^2 \, d\tau\right).\end{aligned}$$

So we have

$$|w|_2^2 + \int_0^t \|\nabla w\|_2^2 \, d\tau \le C\left(|w_0|_2^2 + |\kappa|_\infty^2 + |\sigma|_\infty^2 + \int_0^t \|\nabla \varphi_0\|_2^2 \, d\tau\right). \tag{4.17}$$

To complete the estimate (4.3) we go back to (4.8), which implies by (4.11),

$$\begin{aligned}\|\nabla \varphi\|_2^2 \le C\Big(&\|\nabla \varphi_2\|_q^2 |w|_2^{2(1-(n/q))}\left(|w|_2^2 + \|\nabla w\|_2^2\right)^{n/q} \\ &+ |\sigma|_\infty^2 \|\nabla \varphi_2\|_q^2 + \|\nabla \varphi_0\|_2^2\Big).\end{aligned} \tag{4.18}$$

Integrating between zero and t and applying Hölder's inequality we arrive at

$$\begin{aligned}
&\int_0^t \|\nabla\varphi\|_2^2 \, d\tau \\
&\leq C\left[\left\{\int_0^t \|\nabla\varphi_2\|_q^{2q/(q-n)} \, d\tau\right\}^{(q-n)/q}\left\{\int_0^t |w|_2^2+\|\nabla w\|_2^2 \, d\tau\right\}^{n/q} \sup_{\tau\leq t}|w|_2^{2(1-(n/q))}\right. \\
&\quad \left. +|\sigma|_\infty^2\left\{\int_0^t \|\nabla\varphi_2\|_2^{2q/(q-n)} \, d\tau\right\}^{(q-n)/q}+\int_0^t \|\nabla\varphi_0\|_2^2 \, d\tau\right] \\
&\leq C\left(|w_0|_2^2+|\kappa|_\infty^2+|\sigma|_\infty^2+\int_0^t \|\nabla\varphi_0\|_2^2 \, d\tau\right)
\end{aligned}$$

by (4.17). This completes the proof of (4.3).

Remark 4.1. If $\kappa_i \equiv 1$, Theorem 4.1 holds when we just assume that

$$\nabla\varphi_i \in L^{2q/(q-n)}\left(0,T;L^q\left(\Omega\right)\right), \qquad q>n\vee 2$$

since in the second side of (4.4) the two first integrals disappear.

COROLLARY 4.1. *There exists at most one weak solution to* (1.1) *with the boundary conditions* (1.1b) *or* (1.1b′) *such that*

$$\nabla u, \nabla\varphi \in L^{2q/(q-n)}\left(0,T;L^q\left(\Omega\right)\right), \qquad q>n\vee 2, \tag{4.19}$$

where $n\vee 2$ denotes the maximum of 2 *and* n.

Proof. If $(u_i,\varphi_i), i=1,2$, are two weak solutions to (1.1) with the boundary conditions (1.1b) or (1.1b′) and corresponding to the same initial and boundary data, then (4.3) reads

$$|w(t)|_2^2+\int_0^t \|\nabla w(\tau)\|_2^2 \, d\tau+\int_0^t \|\nabla\varphi(\tau)\|_2^2 \, d\tau \leq 0.$$

and the result follows (see also Remark 4.1).

THEOREM 4.2. *Assume that* (4.1) *holds and that there exists one weak solution* (u_1,φ_1) *to* (1.1) *with the boundary conditions* (1.1b) *or* (1.1b′) *such that*

$$\begin{aligned}
\nabla u_1 \in L^{2q/(q-n)}\left(0,T;L^q\left(\Omega\right)\right), \quad & \nabla\varphi_1 \in L^{4q/(q-n)}\left(0,T;L^q\left(\Omega\right)\right), \\
q>n\vee 2, \quad & \varphi_1 \text{ bounded},
\end{aligned} \tag{4.20}$$

where $n\vee 2$ denotes the maximum of 2 *and* n. *Then, every weak solution* (*or classical solution*) (u_2,φ_2) *to* (1.1), *which is such that φ_2 is bounded, agrees with it.*

Proof. If we set $w=u_1-u_2, \varphi=\varphi_1-\varphi_2$ we have

(4.21)

$$\begin{aligned}
w_t &= \nabla\cdot\left(\kappa(u_2)\nabla w+\left(\kappa(u_1)-\kappa(u_2)\right)\nabla u_1\right)+\nabla\cdot\left(\sigma(u_1)\varphi_1\nabla\varphi_1-\sigma(u_2)\varphi_2\nabla\varphi_2\right) \\
&= \nabla\cdot\left(\kappa(u_2)\nabla w+\left(\kappa(u_1)-\kappa(u_2)\right)\nabla u_1\right)+\nabla\cdot\left(\left(\sigma(u_1)-\sigma(u_2)\right)\varphi_1\nabla\varphi_1\right. \\
&\qquad \left. +\sigma(u_2)\left(\varphi_1-\varphi_2\right)\nabla\varphi_1+\sigma(u_2)\varphi_2\left(\nabla\varphi_1-\nabla\varphi_2\right)\right) \\
&= \nabla\cdot\left(\kappa(u_2)\nabla w+\left(\kappa(u_1)-\kappa(u_2)\right)\nabla u_1\right)+\nabla\cdot\left(\left(\sigma(u_1)-\sigma(u_2)\right)\varphi_1\nabla\varphi_1\right. \\
&\qquad \left. +\sigma(u_2)\varphi\nabla\varphi_1+\sigma(u_2)\varphi_2\nabla\varphi\right).
\end{aligned}$$

If we multiply by w and integrate over Ω we get

$$\frac{1}{2}\frac{d}{dt}|w|_2^2 + \kappa_1 \|\nabla w\|_2^2 \le I_1 + I_2 + I_3 + I_4, \tag{4.22}$$

where

$$I_1 = -\int_\Omega (\sigma(u_1) - \sigma(u_2))\, \varphi_1 \nabla\varphi_1 \cdot \nabla w \, dx,$$
$$I_2 = -\int_\Omega \sigma(u_2)\, \varphi \nabla\varphi_1 \cdot \nabla w \, dx,$$
$$I_3 = -\int_\Omega \sigma(u_2)\, \varphi_2 \nabla\varphi \cdot \nabla w \, dx,$$
$$I_4 = -\int_\Omega (\kappa(u_1) - \kappa(u_2))\, \nabla u_1 \cdot \nabla w \, dx.$$

Since σ and the φ_i's are bounded we obtain, by Hölder's inequality,

$$|I_1| \le C \int_\Omega |\nabla\varphi_1| |\nabla w| |w| \, dx \le C \|\nabla\varphi_1\|_q \|\nabla w\|_2 |w|_{2q/(q-2)}, \tag{4.23}$$

$$|I_2| \le C \int_\Omega |\varphi| |\nabla\varphi_1| |\nabla w| \, dx \le C \|\nabla\varphi_1\|_q \|\nabla w\|_2 |\varphi|_{2q/(q-2)},$$

$$|I_3| \le C \int_\Omega |\nabla\varphi| |\nabla w| \, dx \le C \|\nabla\varphi\|_2 \|\nabla w\|_2 . \tag{4.24}$$

$$|I_4| \le C \int_\Omega |\nabla u_1| |\nabla w| |w| \, dx \le C \|\nabla u_1\|_q \|\nabla w\|_2 |w|_{2q/(q-2)} . \tag{4.25}$$

Since $q > n$ from the Sobolev imbedding theorem, we get

$$|I_2| \le C \|\nabla\varphi_1\|_q \|\nabla w\|_2 \|\nabla\varphi\|_2 . \tag{4.26}$$

Now from the equation satisfied by φ_1, φ_2 we have

$$0 = \nabla \cdot (\sigma(u_2) \nabla\varphi_2) = \nabla \cdot (\sigma(u_2) \nabla(\varphi_2 - \varphi_1)) + \nabla \cdot ((\sigma(u_2) - \sigma(u_1)) \nabla\varphi_1) .$$

Multiplying by φ and integrating on Ω we obtain

$$\int_\Omega \sigma(u_2) |\nabla\varphi|^2 \, dx = \int_\Omega (\sigma(u_1) - \sigma(u_2)) \nabla\varphi_1 \cdot \nabla\varphi \, dx.$$

Hence by Hölder's inequality,

$$\|\nabla\varphi\|_2^2 \le C \int_\Omega |w| |\nabla\varphi_1| |\nabla\varphi| \, dx \le C \|\nabla\varphi\|_2 \|\nabla\varphi_1\|_q |w|_{2q/(q-2)} . \tag{4.27}$$

Collecting (4.22)–(4.27) we get

$$\frac{1}{2}\frac{d}{dt}|w|_2^2 + \kappa_1 \|\nabla w\|_2^2 \le C \left\{ \|\nabla\varphi_1\|_q \|\nabla w\|_2 |w|_{2q(q-2)} + \|\nabla u_1\|_q \|\nabla w\|_2 |w|_{2q/(q-2)} \right.$$
$$\left. + \|\nabla\varphi_1\|_q^2 \|\nabla w\|_2 |w|_{2q/(q-2)} \right\} .$$

Applying Young's inequality we easily deduce that

$$\frac{1}{2}\frac{d}{dt}|w|_2^2 + \frac{\kappa_1}{2} \|\nabla w\|_2^2 \le C \left\{ \|\nabla\varphi_1\|_q^2 + \|\nabla u_1\|_q^2 + \|\nabla\varphi_1\|_q^4 \right\} |w|_{2q/(q-2)}^2 .$$

By (4.11) and again applying Young's inequality we get

$$\frac{1}{2}\frac{d}{dt}|w|_2^2+\frac{\kappa_1}{2}\|\nabla w\|_2^2 \le C\{\|\nabla\varphi_1\|_q^2+\|\nabla u_1\|_q^2+\|\nabla\varphi_1\|_q^4\}|w|_2^{2(1-(n/q))}\cdot(|w|_2^2+\|\nabla w\|_2^2)^{n/q}$$
$$\le \varepsilon(|w|_2^2+\|\nabla w\|_2^2)+C_\varepsilon\left\{\|\nabla\varphi_1\|_q^{2q/(q-n)}+\|\nabla u_1\|_q^{2q/(q-n)}\right.$$
$$\left.+\|\nabla\varphi_1\|_q^{4q/(q-n)}\right\}|w|_2^2.$$

Choosing $\varepsilon=\kappa_1/2$ we obtain

$$\frac{d}{dt}|w|_2^2 \le C\left\{1+\|\nabla u_1\|_q^{2q/(q-n)}+\|\nabla\varphi_1\|_q^{4q/(q-n)}\right\}|w|_2^2.$$

Since

$$1+\|\nabla u_1\|_q^{2q/(q-n)}+\|\nabla\varphi_1\|_q^{4q/(q-n)}\in L^1(0,T),$$

the result follows from the Gronwall inequality.

Remark 4.2. These results improve preceding results of [Ch.C.]. Note that (4.2) holds automatically when $n=1$; see [Ch.C.].

5. A blowup result. The results of this section improve and complete the results contained in [A.C.1]. Interesting results on blowup could also be found in [L].

Let us consider $(u(x,t),\varphi(x,t))$, a local solution to

$$\begin{aligned}
&u_t=\nabla\cdot(\kappa(u)\nabla u)+\sigma(u)|\nabla\varphi|^2, \quad x\in\Omega, \quad t>0,\\
&\partial u/\partial n=0, \quad x\in\Gamma, \quad t>0,\\
&u(x,0)=u_0(x), \qquad x\in\Omega,\\
&\nabla\cdot(\sigma(u)\nabla\varphi)=0, \quad x\in\Omega, \quad t>0,\\
&\varphi=\varphi_0, \quad x\in\Gamma, \quad t>0.
\end{aligned} \tag{5.1}$$

Let us assume that

$$u_0(x)\ge 0, \qquad x\in\Omega. \tag{5.2}$$

$$0<\kappa(s), \quad \sigma(s)<+\infty \quad \forall s\ge 0, \quad \sigma \text{ differentiable}, \quad \sigma'(s)\ge 0 \quad \forall s\ge 0, \tag{5.3}$$

$$\int_0^{+\infty}\frac{ds}{\sigma(s)}<+\infty. \tag{5.4}$$

If $d\gamma(x)$ is the superficial measure on Γ we remark that

$$\lambda\to\int_\Gamma|\varphi-\lambda|^2\,d\gamma(x)$$

achieves its minimum value for

$$\lambda=\bar\varphi=\frac{1}{|\Gamma|}\int_\Gamma\varphi\,d\gamma(x).$$

So, if we set

$$\varphi_\Omega = \frac{1}{|\Omega|} \int_\Omega \varphi \, dx,$$

we have, for some constant C,

$$\int_\Gamma |\varphi - \bar{\varphi}|^2 \, d\gamma(x) \le \int_\Gamma |\varphi - \varphi_\Omega|^2 \, d\gamma(x) \le C \int_\Omega |\nabla \varphi|^2 \, dx \quad \forall \varphi \in H^1(\Omega).$$

Let us denote again by C the best constant such that

$$\int_\Gamma |\varphi - \bar{\varphi}|^2 \, d\gamma(x) \le C \int_\Omega |\nabla \varphi|^2 \, dx \quad \forall \varphi \in H^1(\Omega). \tag{5.5}$$

Then we can prove the following.

THEOREM 5.1. *Assume that*

$$\int_\Omega \int_{u_0(x)}^{+\infty} \frac{ds}{\sigma(s)} \, dx < \frac{1}{C} \int_0^{+\infty} \int_\Gamma |\varphi_0 - \bar{\varphi}_0|^2 \, d\gamma(x) \, dt, \tag{5.6}$$

where

$$\bar{\varphi}_0 = \frac{1}{|\Gamma|} \int_\Gamma \varphi_0 \, d\gamma(x),$$

then (5.1) *cannot have a smooth global solution.*

Proof. Let us assume that (5.1) has a smooth global solution. Define

$$Y(t) = \int_\Omega \left(\int_{u(x,t)}^{+\infty} \frac{ds}{\sigma(s)} \right) dx. \tag{5.7}$$

From (5.2) and the maximum principle (see [F.]) it is clear that

$$u(x,t) \ge 0, \quad x \in \Omega, \quad t > 0, \tag{5.8}$$

and thus $Y(t)$ makes sense and is nonnegative (see (5.4)).

Differentiating we obtain, using (5.1),

$$\begin{aligned} \frac{dY(t)}{dt} &= -\int_\Omega \frac{u_t}{\sigma(u)} \, dx \\ &= -\int_\Omega \frac{\nabla \cdot (\kappa(u) \nabla u) + \sigma(u) |\nabla \varphi|^2}{\sigma(u)} \, dx \\ &= -\int_\Omega \nabla \cdot (\kappa(u) \nabla u) \cdot \frac{1}{\sigma(u)} \, dx - \int_\Omega |\nabla \varphi|^2 \, dx. \end{aligned} \tag{5.9}$$

Integrating by parts we have, since $\partial u / \partial n = 0$ on Γ and by (5.3),

$$-\int_\Omega \nabla \cdot (\kappa(u) \nabla u) \cdot \frac{1}{\sigma(u)} \, dx = -\int_\Omega \frac{\kappa(u) \sigma'(u)}{\sigma^2(u)} |\nabla u|^2 \, dx \le 0. \tag{5.10}$$

Hence

$$\frac{dY(t)}{dt} \leq -\int_\Omega |\nabla\varphi|^2\, dx, \tag{5.11}$$

from which it follows that

$$\frac{dY(t)}{dt} \leq -\frac{1}{C}\int_\Gamma |\varphi_0 - \bar{\varphi}_0|^2\, d\gamma(x).$$

Integrating between zero and t we get

$$0 \leq Y(t) \leq T(0) - \frac{1}{C}\int_0^t \int_\Gamma |\varphi_0 - \bar{\varphi}_0|^2\, d\gamma(x)\, dt,$$

which by (5.6) is impossible for t large.

Remark 5.1. In the case where

$$\varphi_0 = \varphi_0(x), \tag{5.12}$$

it is shown that (5.1) has a global solution if and only if

$$\varphi_0 = \text{ Const.}$$

Indeed, in this case (5.6) holds except when

$$\varphi_0 = \bar{\varphi}_0 = \text{ Const.}$$

A more convincing example showing the sharpness of (5.6) under the assumptions of Theorem 5.1 is the following. Consider $\Omega = (0,1)$. Then if φ is a function in $H^1(0,1)$,

$$\bar{\varphi} = \frac{1}{|\Gamma|}\int_\Gamma \varphi\, d\gamma(x) = \frac{1}{2}\{\varphi(0) + \varphi(1)\}.$$

Moreover,

$$\int_\Gamma |\varphi - \bar{\varphi}|^2\, d\gamma(x) = |\varphi(0) - \bar{\varphi}|^2 + |\varphi(1) - \bar{\varphi}|^2 = \frac{|\varphi(0) - \varphi(1)|^2}{2}.$$

Now we have

$$|\varphi(0) - \varphi(1)| = \left|\int_0^1 \varphi'(s)\, ds\right| \leq \left\{\int_0^1 (\varphi'(s))^2\, ds\right\}^{1/2},$$

which shows by squaring that

$$\frac{|\varphi(0) - \varphi(1)|^2}{2} = \int_\Gamma |\varphi - \bar{\varphi}|^2\, d\gamma(x) \leq \frac{1}{2}\int_0^1 (\varphi'(s))^2\, ds. \tag{5.13}$$

The constant $\frac{1}{2}$ in (5.13) is the best possible as it can be seen by taking

$$\varphi(s) = s.$$

Then consider the one-dimensional version of (5.1) with

$$u(0) = u_0 = \text{ Const.},$$

and look for a solution

$$u = u(t)$$

depending on t only. Set

$$\varphi_0(0,t) = A_0(t), \qquad \varphi_0(1,t) = A_1(t).$$

Then, clearly, the equation satisfied by φ leads to

$$\varphi(x,t) = A_0(t) + x(A_1(t) - A_0(t)),$$

and the equation in u becomes

$$u_t = \sigma(u)(A_1(t) - A_0(t))^2 \tag{5.14}$$

or

$$\int_{u_0}^{u} \frac{ds}{\sigma(s)} = \int_0^t (A_1(s) - A_0(s))^2\, ds.$$

In the case we are considering, the failure of (5.6) reads

$$\int_{u_0}^{+\infty} \frac{ds}{\sigma(s)} \geq \int_0^{+\infty} (A_1(s) - A_0(s))^2\, ds.$$

This implies that (5.14) has a global solution which is bounded when

$$\int_{u_0}^{+\infty} \frac{ds}{\sigma(s)} > \int_0^{+\infty} (A_1(s) - A_0(s))^2\, ds,$$

and is unbounded otherwise.

Remark 5.2. In dimension 1 and when $\kappa \equiv 1$ it is possible to show that $u(x,t)$ blows up globally, i.e., if t^* denotes the blowup time then

$$u(x,t) \to +\infty \quad \text{a.e. } x \in \Omega \quad \text{when } t \to t^*.$$

Indeed, if for instance $\Omega = (0,1)$, then by integrating the equation

$$(\sigma(u)\varphi') = 0$$

we get

$$\sigma(u)\varphi' = C(t).$$

Hence

$$\varphi' = \frac{C(t)}{\sigma(u)}$$

with

$$\varphi(1,t)-\varphi(0,t)=C(t)\int_0^1\frac{dx}{\sigma(u(x,t))}.$$

Setting $\lambda(t)=\varphi(1,t)-\varphi(0,t)$ the equation satisfied by u reads

$$u_t=u_{xx}+\frac{\lambda^2}{\sigma(u)}\left(\int_0^1\frac{dx}{\sigma(u(x,t))}\right)^{-2}.$$

Differentiating in x we see that $v=u_x$ satisfies

$$v_t=v_{xx}-\lambda^2\frac{\sigma'(u)}{\sigma(u)^2}\left(\int_0^1\frac{dx}{\sigma(u(x,t))}\right)^{-2}v,$$

$$v(x,t)=0,\quad x=0,1,\quad v(x,0)=(u_0)_x.$$

Assuming that $(u_0)_x\in L^\infty(0,1)$ it follows from the maximum principle, recall that

$$\lambda^2\,\frac{\sigma'(u)}{\sigma(u)^2}\left(\int_0^1\frac{dx}{\sigma(u(x,t))}\right)^{-2}\geq 0,$$

that

$$|u_x|_\infty\leq|(u_0)_x|_\infty. \tag{5.15}$$

Hence

$$u(x,t)=\int_{x_0}^x u_x(x,t)\,dx+u(x_0,t).$$

If $u(x_0,t)$ blows up, then $u(x,t)$ blows up for any x since the integral is bounded thanks to (5.15).

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